# Exact Results on $\mathcal{O}(\alpha)$ Corrections to the Single Hard Bremsstrahlung Process in Low Angle Bhabha Scattering in the SLC/LEP Energy Regime*

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**Abstract**

We present the exact $\mathcal{O}(\alpha)$ correction to the process $e^+e^- \rightarrow e^+e^- + \gamma$ in the low angle luminosity regime at SLC/LEP energies. We give explicit formulas for the completely differential cross section. As an important application, we illustrate the size of the respective corrections of $\mathcal{O}(\alpha^2)$ to the SLC/LEP luminosity cross section. We show explicitly that our results have the correct infrared limit, as a cross-check. Some comments are made about the implementation of our results in the framework of a Monte Carlo event generator. This latter implementation will appear elsewhere.

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# 1 Introduction

Recently, new luminometers at LEP[1] have made measurements of the luminosity process $e^+e^- \rightarrow e^+e^- + n(\gamma)$ at the experimental precision tags below .1%. This should be compared with the prediction by two of us (S.J. and B.F.L.W.), with E. Richter-Wąs and Z. Wąs, of these processes at the .25% precision tag in Ref. [2] using the YFS Monte Carlo event generator BHLUMI2.00. Recently, using version 4.0 of BHLUMI, the authors in Ref. [2] have reported precision tags $\sim$ .1% for the ALEPH SICAL detector's asymmetric acceptance theoretical cross section prediction [3, 4] and .16% precision for the general ALEPH SICAL acceptance. The four of us, with W. Płaczek, E. Richter-Wąs, M. Skrzypek and Z. Wąs [5], have reported the entirely equivalent conservative theoretical precision tag $\sim$ .15% for the bremsstrahlung correction for the general acceptance of the ALEPH SICAL detector. These results are currently being extended to the other new LEP luminometers[6]. It is clear that the precision tag on the theoretical prediction of $\sigma_{\mathcal{L}}$, the SLC/LEP luminosity cross section, urgently needs further improvement over the $\sim$ .15% level if the theoretical uncertainties are going to be reduced to the required level of one half of $\Delta\sigma_{\mathcal{L}}^{\rm exp}$, the respective experimental uncertainty, so that they do not obscure the comparison between theory and experiment in the high precision $Z^0$ physics tests of the Standard Model of the electroweak interaction.

From Table 2 in Ref. [2] and from Table 1 in Ref. [3], we see that the subleading part of the $\mathcal{O}(\alpha^2)$ pure bremsstrahlung correction to $\sigma_{\mathcal{L}}$ remains, at this writing, a dominant part of the outstanding theoretical uncertainty, where it contributes directly to the physical precision and indirectly to the technical precision both at a level $\sim$ .1% itself. Exact results on the double bremsstrahlung process itself have been derived in Refs. [7] and [8] and, in fact, the entire leading log part of the $\mathcal{O}(\alpha^2)$ correction to $\sigma_{\mathcal{L}}$ has recently [9] been incorporated into BHLUMI4.0. What remains to be done rigorously then is to compute the remaining exact subleading part of the bremsstrahlung correction to $\sigma_{\mathcal{L}}$ and incorporate it into our YFS Monte Carlo event generator

BHLUMI4.xx to check precisely the size of this effect in $\sigma_{\mathcal{L}}$.

In this paper, we present the exact results which are necessary to evaluate this remaining unquantified sub-leading part of the bremsstrahlung correction to $\sigma_{\mathcal{L}}$ with particular emphasis on the acceptances of the new LEP high precision luminometers. While partial results and estimates on this sub-leading correction to $\sigma_{\mathcal{L}}$ have appeared elsewhere[10, 11], our results are the first ever, fully differential exact results of their type. Their detailed implementation into BHLUMI4.xx will appear elsewhere[6].

More precisely, the lone outstanding contribution from bremsstrahlung processes to $\sigma_{\mathcal{L}}$ which dominates the physical precision part of the error in BHLUMI4.0 from $\mathcal{O}(\alpha^2)$ bremsstrahlung is the sub-leading part of the virtual correction to the single bremsstrahlung process. Thus, it is this latter process which we shall compute exactly in what follows in a fully differential manner, as is needed for Monte Carlo event generator applications. We repeat — such a completely differential, exact $\mathcal{O}(\alpha^2)$ single bremsstrahlung calculation has not appeared elsewhere. (See, however, Refs. [10, 11] for various levels of partial results.)

Our work is organized as follows. In Sect. 2, we set our kinematic and notational conventions. In Sect. 3, we analyze the processes of interest to us using the algebraic program FORM [12]. In Sect. 4, we present numerical results which illustrate checks on our work in the SLC/LEP luminosity regime. Sect. 5 contains our summary remarks. The Appendices contain some technical details.

# 2 Preliminaries

In this section we set our kinematical notation and calculational conventions. We begin with the kinematics.

The process under discussion is illustrated in Fig. 1. We consider the one-loop corrections to the process $e^+(p_1) + e^-(q_1) \rightarrow e^+(p_2) + e^-(q_2) + \gamma(k)$ in the low angle regime of the SLC/LEP luminometers, where $\theta_{e^+}, \theta_{e^-} \in [25\ \text{mrad}, 70\ \text{mrad}]$ if $\theta_{e^+}, \theta_{e^-}$ are the CMS scattering angles of $e^+$, $e^-$ in

the $Z^0$ resonance energy regime. The kinematics is illustrated in Fig. 1. It can be shown [13, 10] that, in this low angle regime, graphs involving the exchange of more than one virtual photon line between different fermion lines are suppressed in the $\mathcal{O}(\alpha^2)$ correction to the cross section. Thus, we do not need to calculate these graphs for the $\mathcal{O}(\alpha^2)$ corrections of interest to us here. Further, it can be shown that those terms in the cross section involving interference of the radiation from the $e^+$ line with that from the $e^-$ line, in the low-angle Bhabha scattering (LABH) regime, are severely suppressed as well — this is the so-called up-down interference suppression [13]. Hence, we will only need to calculate the graphs, as those shown in Fig. 1, where only one photon is exchanged between $e^+$ and $e^-$ lines and in which a virtual correction exists on one of these lines. There are a total of 36 such graphs, excluding vacuum polarization, and the ten electron-line emission graphs giving non-trivial results in our on-shell renormalization scheme are shown. The associated positron-line emission graphs must be calculated as well. The $s$-channel exchange graphs will also be calculated for completeness and we will see that they are indeed negligible at the level of accuracy of interest to us here.

For the actual calculation, we rely on two technical tools. We evaluate the general $\gamma$ emission amplitudes using the formulation of the CALKUL [14, 15] methods given by Xu *et al.* [16]. Thus, our massless fermion spinors $|p, \pm\rangle$, of helicity $\pm$, are such that their spinor product is

$$\begin{aligned} \langle p|p'\rangle_+ &\equiv \langle p, -|p', +\rangle = (\mathbf{p}_x + i\mathbf{p}_y)\frac{\sqrt{p'_+}}{\sqrt{p_+}} - (\mathbf{p}'_x + i\mathbf{p}'_y)\frac{\sqrt{p_+}}{\sqrt{p'_+}} \qquad (2.1) \\ \langle p|p'\rangle_- &= \langle p'|p\rangle_+^* \end{aligned}$$

for $p_+ = p^0 + \mathbf{p}_z$ in an obvious cartesian coordinate notation for the 3-momentum $\mathbf{p}$ (i.e. $p^\mu = (p^0, p^1, p^2, p^3) = (p^0, \mathbf{p}) = (p^0, p_x, p_y, p_z)$), and our corresponding photon polarization vectors are, for helicity $\rho$,

$$\epsilon_\mu(k, h, \rho) = \frac{\rho}{\sqrt{2}} \frac{\langle h, -\rho|\gamma_\mu|k, -\rho\rangle}{\langle h|k\rangle_\rho}\ , \qquad (2.2)$$

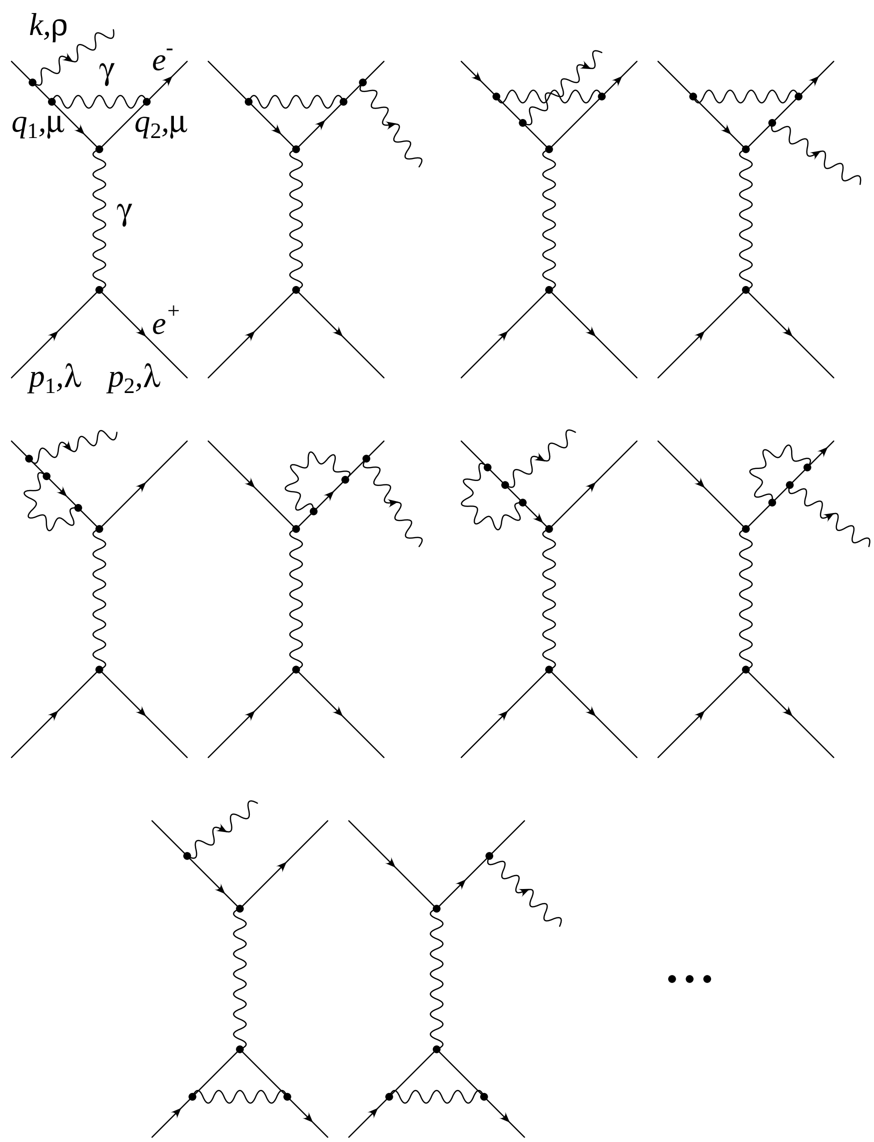


Figure 1: $\mathcal{O}(\alpha^2)$ single bremsstrahlung correction in $e^+e^- \to e^+e^-$ at low angles. Only electron line emission graphs are shown.

where $k$ is the photon momentum and $h$ is an auxiliary massless 4-vector. As usual, $h$ may be chosen to simplify a given gauge-invariant set of graphs[16]. For a given helicity, three of the ten graphs in Fig. 1 can be eliminated in this manner, leaving seven to be evaluated. The methods of Xu *et al.* are augmented for the evaluation of virtual corrections by the algebraic manipulation program FORM[12], which we use to evaluate our typical one-loop integrals via reduction to the now-standard scalar integrals; here we use the realization of the standard scalar integrals of Refs. [17, 18], since they were numerically stable enough for our applications. It was with the aid of these two computational techniques that we have calculated our results for the $\mathcal{O}(\alpha)$ correction to the single bremsstrahlung process $e^+e^- \to e^+e^- + \gamma$ at low SLC/LEP luminosity regime angles. Our results are presented in the next section.

# 3 Exact $\mathcal{O}(\alpha^2)$ Results for $\mathbf{e^+e^- \to e^+e^- + \gamma}$

In this section we present our results for the $\mathcal{O}(\alpha)$ correction to the process $e^+e^- \to e^+e^- + \gamma$ at low angles. We discuss first the $2\gamma$ bremsstrahlung effect.

For our $2\gamma$ bremsstrahlung correction, we use the results of three of us (S.J., B.F.L.W, and S.A.Y.) in Ref. [8], where the exact result for $e^+e^- \to e^+e^- + 2\gamma$ was computed with the methods of Xu *et al.* [16]. These results have been checked in Ref. [19].

For the virtual correction to $e^+e^- \to e^+e^- + \gamma$, we organize our results in terms of the amplitude for $\gamma$ emission from the electron line and for emission from the positron line, neglecting the so-called up-down interference terms [13], since these are known to contribute negligibly at the level of our current precision of interest.

The electron-line emission amplitude with one virtual photon is

$$A^{e^-}_{(1)} = \frac{ie^5}{16\pi^2 t_p} \left(\mathcal{F}_0\mathcal{I}_0 + \mathcal{F}_1\mathcal{I}_1 + \mathcal{F}_2\mathcal{I}_2\right) \tag{3.3}$$

where

$$\mathcal{I}_0 \;=\; 2\sqrt{2}\rho\frac{\langle p_1,p_2\rangle_{-\rho}\left(\langle q_j,p_i\rangle_\rho\right)^2}{\langle q_1,k\rangle_\rho\langle q_2,k\rangle_\rho}\;, \tag{3.4}$$

$$\mathcal{I}_1 \;=\; 2\sqrt{2}\mu\frac{\langle q_{\hat{\jmath}},k\rangle_{-\rho}\langle p_2,q_j\rangle_{-\lambda}\langle q_j,p_1\rangle_\lambda}{\langle q_{\hat{\jmath}},k\rangle_\rho\langle q_1,q_2\rangle_{-\rho}}\;, \tag{3.5}$$

$$\mathcal{I}_2 \;=\; 2\sqrt{2}\mu\frac{\langle q_{\hat{\jmath}},k\rangle_{-\rho}\langle p_2,q_{\hat{\jmath}}\rangle_{-\lambda}\langle q_{\hat{\jmath}},p_1\rangle_\lambda}{\langle q_{\hat{\jmath}},k\rangle_\rho\langle q_1,q_2\rangle_{-\rho}}\;, \tag{3.6}$$

where the helicity-dependent indices $i,j,\hat{\jmath}$ are given by

$$i=\left\{\begin{array}{l}1\\2\end{array}\right.\text{ if }\rho=\pm\lambda,\qquad (j,\hat{\jmath})=\left\{\begin{array}{l}(1,2)\\(2,1)\end{array}\right.\text{ if }\rho=\pm\mu, \tag{3.7}$$

where $\lambda,\mu,\rho$ are the helicities of the positron, electron, and photon. The function $\mathcal{I}_0$ is proportional to the electron line Born amplitude:

$$A^{\mathrm{e}^-}_{\mathrm{Born}}=\frac{ie^3}{t_p}\,\mathcal{I}_0\;. \tag{3.8}$$

The form factors are

$$\begin{aligned}
\mathcal{F}_0(\rho=\mu) &= -8-8m_e^2C_{123}^{[0]}+r_1(t_q-r_1)^{-1}-2(t_qC_{124}+t_pC_{134}^{[0]})\\
&- \;\{t_qC_{124}\;-r_1C_{123}^{[r_1]}\;-(t_q+r_2)C_{134}^{[r_1]}\;+(r_1-r_2)C_{234}\;+r_1t_qD_{1234}^{[r_1]}\;\}\\
&\qquad\qquad\times\; t_qr_2^{-1}(r_1-r_2)(t_q-r_1)^{-1}\\
&+ \;\{t_qC_{124}\;+r_2C_{123}^{[-r_2]}\;-(t_q-r_1)C_{134}^{[-r_2]}\;+(r_1-r_2)C_{234}\;-r_2t_qD_{1234}^{[-r_2]}\;\}\\
&+ \;6B_{12}\;+\;(B_{13}^{[r_1]}-B_{34})\;r_1(t_q-r_1)^{-1}\;\{1-3t_q(t_q+r_2)^{-1}\}\\
&- \;6B_{34}\;+\;(B_{24}-B_{34})\;\{2t_qr_1(r_1-r_2)^{-1}(t_q-r_1)^{-1}\},
\end{aligned} \tag{3.9}$$

$$
\begin{aligned}
\mathcal{F}_1(\rho=\mu) &= 2t_q(r_1-r_2)^{-1} - t_q(t_q-r_1)^{-1} \\
&+ \ \{t_q C_{124} \ - r_1 C_{123}^{[r_1]} \ - (t_q+r_2) C_{134}^{[r_1]} \ + (r_1-r_2) C_{234} \ + r_1 t_q D_{1234}^{[r_1]} \ \} \\
&\qquad\qquad \times \{t_p t_q r_2^{-2} (t_q-r_1)^{-1}(t_q-r_2) + \tfrac{1}{2}\delta_{\rho,1}\} \\
&- \ \{t_q C_{124} \ + r_2 C_{123}^{[-r_2]} \ - (t_q-r_1) C_{134}^{[-r_2]} \ + (r_1-r_2) C_{234} \ - r_2 t_q D_{1234}^{[-r_2]} \ \} \\
&\qquad\qquad \times \{\tfrac{1}{2} r_1^{-1} r_2 \delta_{\rho,-1}\} \\
&+ \ (B_{13}^{[r_1]} - B_{34}) \ t_p t_q (t_q-r_1)^{-1} \ \{2r_2^{-1} - 3(t_q+r_2)^{-1}\} \\
&+ \ 2(B_{24} - B_{34}) \ t_p t_q (r_1-r_2)^{-1} \{(r_1-r_2)^{-1} - t_q r_2^{-1} (t_q-r_1)^{-1}\},
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
\mathcal{F}_2(\rho=\mu) &= -2t_q(r_1-r_2)^{-1} + t_q(t_q+r_2)^{-1} \\
&- \ \{t_q C_{124} \ - r_1 C_{123}^{[r_1]} \ - (t_q+r_2) C_{134}^{[r_1]} \ + (r_1-r_2) C_{234} \ + r_1 t_q D_{1234}^{[r_1]} \ \} \\
&\qquad\qquad \times \{t_p t_q r_2^{-2} + \tfrac{1}{2} r_1 r_2^{-1} \delta_{\rho,1}\} \\
&+ \ \{t_q C_{124} \ + r_2 C_{123}^{[-r_2]} \ - (t_q-r_1) C_{134}^{[-r_2]} \ + (r_1-r_2) C_{234} \ - r_2 t_q D_{1234}^{[-r_2]} \ \} \\
&\qquad\qquad \times \{\tfrac{1}{2} \delta_{\rho,-1}\} \\
&+ \ (B_{34} - B_{13}^{[r_1]}) \ t_p t_q (t_q+r_2)^{-1} \ \{2r_2^{-1} + (t_q+r_2)^{-1}\} \\
&+ \ 2(B_{24} - B_{34}) \ t_p t_q (r_1-r_2)^{-1} \ \{r_2^{-1} - (r_1-r_2)^{-1}\}.
\end{aligned}
\tag{3.11}
$$

The opposite helicity cases may be obtained from the above results using the substitutions (for $i = 0, 1, 2$)

$$
\mathcal{F}_i(\rho=-\mu, r_1, r_2) = \mathcal{F}_i(\rho=\mu, -r_2, -r_1) \ . \tag{3.12}
$$

The scalar integrals $B$, $C$, $D$ are defined in Refs. [17] (we evaluate them using algorithms from Refs. [17]) and in Appendix A. The kinematic variables $s, s', t_p, t_q, r_i$ are defined as

$$
\begin{aligned}
s &= (p_1+q_1)^2, \qquad s' = (p_2+q_2)^2, \\
t_p &= (p_1-p_2)^2, \qquad t_q = (q_1-q_2)^2, \\
r_i &= 2 q_i \cdot k.
\end{aligned}
\tag{3.13}
$$

We have found that at low angles in the SLC/LEP energy regime, the $\mathcal{F}_0\mathcal{I}_0$ terms in (3.3) are often a good approximation to the entire result, and that these terms are in turn well approximated by the simple expression

$$A^{e^-}_{(1)\ \mathrm{approx}} = \frac{ie^5}{16\pi^2 t_p}\, \mathcal{F}_0^{\ \mathrm{approx}} \quad \mathcal{I}_0\ , \tag{3.14}$$

$$\begin{aligned}
\mathcal{F}_0^{\ \mathrm{approx}} &= -8 - 8m_e^2 C^{[0]}_{123} - 2(t_q C_{124} + t_p C^{[0]}_{134}) + 6B_{12} - 6B_{34} \\
&= -8 + \frac{2\pi^2}{3} - \ln^2\frac{|t_p|}{m_e^2} - \ln^2\frac{|t_q|}{m_e^2} + 6\ln\frac{|t_p|}{m_e^2} \\
&\quad + 4\ln\frac{|t_p|}{m_e^2}\ln\frac{m_\gamma}{m_e} + 4\ln\frac{|t_q|}{m_e^2}\ln\frac{m_\gamma}{m_e} - 8\ln\frac{m_\gamma}{m_e}.
\end{aligned} \tag{3.15}$$

This approximation has been compared to the complete result in detail and we have found it to be within 10% of the result (3.3) throughout most of the final particle phase space; such comparisons will be presented in more detail elsewhere[6].

The analog of (3.3) for positron line emission is obtained by crossing:

$$A^{e^+}_{(1)} = A^{e^-}_{(1)}\ (p_1 \leftrightarrow -q_2', q_1 \leftrightarrow -p_2, \lambda \leftrightarrow -\mu). \tag{3.16}$$

The differential cross section associated with (3.3) is the usual

$$\frac{d\sigma^{\mathcal{O}(\alpha^2)}}{d\Omega k dk d\Omega_k} = \frac{(p_2{}^0)^2}{(4\pi)^5 s s'} \sum_{\lambda,\mu,\rho} \mathrm{Re}\ \left(A^{e^+}_{(1)} + A^{e^-}_{(1)}\right)\left(A^{e^+}_{\mathrm{Born}} + A^{e^-}_{\mathrm{Born}}\right)^*, \tag{3.17}$$

where $\Omega$ is the outgoing positron solid angle in the lab frame, $\Omega_k$ is the photon solid angle, and the up-down interference terms can be neglected here.

We have used massless fermion spinors to calculate (3.3) – (3.17). The relevant mass effects can be restored by the standard methods already published in Ref. [15]. We have done that, and it amounts to adding an $m_e^2$-correction term for each external emission line in Fig. 1. The corresponding corrections are given in the Appendix B for completeness. In this way, we have arrived at an exact $\mathcal{O}(\alpha)$ correction to the process $e^+e^- \to e^+e^- + \gamma$ with all relevant mass effects taken explicitly into account.

# 4 Results and Checks

Our results (3.3) – (3.17) are readily introduced into the Monte Carlo program BHLUMI [2, 4] of two of us. This will be presented elsewhere[6]. Here, we wish to discuss some numerical checks we have made on these results.

For our checks, we define an IR-regular differential cross section by subtracting the virtual infrared contribution as given by the YFS theory[20]:

$$\frac{d\sigma^{\mathcal{O}(\alpha^2)}_{\mathrm{IR-reg}}}{d\Omega k dk d\Omega_k} = \frac{d\sigma^{\mathcal{O}(\alpha^2)}}{d\Omega k dk d\Omega_k} - 2\alpha(\mathrm{Re}\, B_{\mathrm{YFS}}) \frac{d\sigma^{\mathcal{O}(\alpha)}}{d\Omega k dk d\Omega_k} , \tag{4.18}$$

with $B_{\mathrm{YFS}}$ defined as in Ref. [22]. We have verified that our infrared limit $k \to 0$ is correct and in agreement with the YFS theory[20] of that limit. This agreement with YFS theory is further illustrated in Fig. 2, where we show that (4.18) is independent of our photon regulator mass $m_\gamma$.

Further, looking into the results in Fig. 2, we see, in addition to their independence of $\ln m_\gamma$, that the size of our $\mathcal{O}(\alpha^2)$ virtual correction is $\sim 0.05$ of the respective real $\mathcal{O}(\alpha)$ correction itself, which is consistent with the naive LL power counting expectations, since $\frac{\alpha}{\pi}L \sim 0.05$ here. Illustrations of the actual size of the subleading $\alpha^2 L$ part of the exact result (3.3) in the ALEPH SICAL-type acceptance in connection with BHLUMI4.xx will appear elsewhere[6].

For completeness, we note that we have also evaluated the exact result for the s-channel exchange contribution to the $\mathcal{O}(\alpha^2)$ correction to the cross sections evaluated here. We find that it is below 0.04 of the $\mathcal{O}(\alpha^2)$ correction itself. This shows that such exchanges are indeed negligible at the level of precision of interest to us here in the evaluation of our respective $\mathcal{O}(\alpha^2)$ effects.

Thus, the important exact $\mathcal{O}(\alpha)$ correction to $e^+e^- \to e^+e^- + \gamma$ has now been calculated and it has been verified that it has the correct IR limit as well as a size consistent with the naive power counting expectations. The implementation of this exact result into the Monte Carlo event generator BHLUMI4.xx [9, 4] for LABH is now in progress and will appear elsewhere[6].

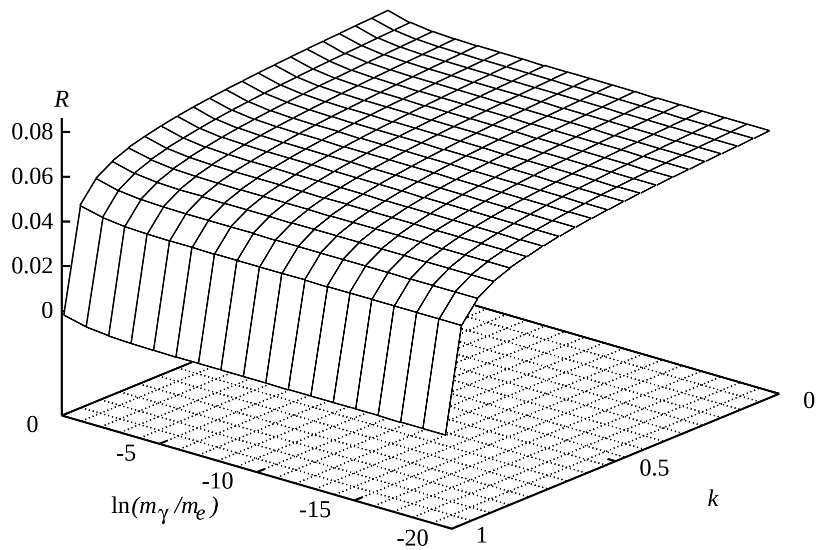


Figure 2: $\ln m_\gamma$ dependence of the ratio $R = \frac{d\sigma^{\mathcal{O}(\alpha^2)}_{\mathrm{IR-reg}}}{d\Omega k dk d\Omega_k} \Big/ \frac{d\sigma^{\mathcal{O}(\alpha)}}{d\Omega k dk d\Omega_k}$ where the numerator is defined in (4.18) and the denominator is the $\mathcal{O}(\alpha)$ single bremsstrahlung cross-section. We plot $R$ as a function of $k = 1 - s'/s$ and $\ln(m_\gamma/m_e)$ for positron polar angles $(\theta, \phi) = (2^\circ, 0^\circ)$ and photon polar angles $(\theta, \phi) = (1.5^\circ, 10^\circ)$ with respect to the positron beam axis. This shows that $\ln m_\gamma$ has cancelled out when $m_\gamma \ll m_e$, so that the result is independent of the infrared regulator.

# 5 Conclusion

In this paper, we have presented the first complete exact $\mathcal{O}(\alpha)$ correction to $e^+e^- \rightarrow e^+e^- + \gamma$ at low angles in the SLC/LEP energy regime. We have checked the infrared limit of our work against known expectations. We have evaluated our results in the ALEPH SICAL type acceptance and found that the respective correction is consistent in size with the naive power counting expectations for its LL content. Implementation of our results into the Monte Carlo event generator BHLUMI [4] for LABH is in progress and will appear elsewhere[6].

We should note that the authors in Ref. [10] have also given pioneering results for this $\mathcal{O}(\alpha)$ correction to $e^+e^- \rightarrow e^+e^- + \gamma$ at low angles. We have not compared our results with theirs however because they have not provided us with a final version of their fully differential results to date. We await their final published expressions with which we may compare.

In summary, the dominant missing contribution to the $\mathcal{O}(\alpha^2)$ bremsstrahlung effect in the theoretical prediction for the SLC/LEP luminosity process $e^+e^- \rightarrow e^+e^-$, the $\mathcal{O}(\alpha^2 L)$ correction, has now been calculated exactly. We look forward with excitement to the application of this correction to reach a new level of precision on the respective theoretical prediction, via BHLUMI4.xx [4], of this luminosity. Such applications will appear elsewhere[6].

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# Appendices

# A Scalar Integrals

In this appendix, we define the scalar integrals used in the formulas in the text. We have, from Ref. [17], with the kinematic conventions (3.14),

$$
\begin{aligned}
B_{12} &= B(m_e{}^2; m_\gamma, m_e) && \text{(A.19)}\\
B_{13}^{[r]} &= B(m_e{}^2 - r; m_\gamma, m_e) && \text{(A.20)}\\
B_{24} &= B(t_q; m_e, m_e) && \text{(A.21)}\\
B_{34} &= B(t_p; m_e, m_e) && \text{(A.22)}\\
C_{123}^{[r]} &= C(m_e{}^2, m_\gamma{}^2, m_e{}^2 - r; m_\gamma, m_e, m_e) && \text{(A.23)}\\
C_{124} &= C(m_e{}^2, t_q, m_e{}^2; m_\gamma, m_e, m_e) && \text{(A.24)}\\
C_{134}^{[r]} &= C(m_e{}^2 - r, t_p, m_e{}^2; m_\gamma, m_e, m_e) && \text{(A.25)}\\
C_{234} &= C(m_\gamma{}^2, t_p, t_q; m_e, m_e, m_e) && \text{(A.26)}\\
D_{1234}^{[r]} &= D(m_e{}^2, m_\gamma{}^2, t_p, m_e{}^2, m_e{}^2 - r, t_q; m_\gamma, m_e, m_e, m_e) && \text{(A.27)}
\end{aligned}
$$

where we have defined the basic scalar integrals as

$$
B(p^2, m_1, m_2) = \frac{(2\pi\mu)^{4-D}}{\pi^2 i} \int \frac{d^D q}{(q^2 - m_1{}^2 + i\epsilon)((q+p)^2 - m_2^2 + i\epsilon)} \; , \tag{A.28}
$$

$$
C(p_1^2, p_2^2, (p_1+p_2)^2; m_1, m_2, m_3) = \\
\frac{1}{\pi^2 i} \int \frac{d^4 q}{(q^2 - m_1{}^2 + i\epsilon)((q+p_1)^2 - m_2^2 + i\epsilon)((q+p_1+p_2)^2 - m_3^2 + i\epsilon)} \; ,
$$

$$
D(p_1^2, p_2^2, p_3^2, (p_1+p_2+p_3)^2, (p_1+p_2)^2, (p_2+p_3)^2; m_1, m_2, m_3, m_4) = \tag{A.29}
$$

$$
\frac{1}{\pi^2 i}\int d^4q \; \frac{1}{(q^2-m_1{}^2+i\epsilon)((q+p_1)^2-m_2^2+i\epsilon)}
$$

$$
\times \frac{1}{((q+p_1+p_2)^2-m_3^2+i\epsilon)((q+p_1+p_2+p_3)^2-m_4^2+i\epsilon)} \; . \tag{A.30}
$$

The $B$ integral is defined using dimensional regularization, and the $C$ and $D$ integrals are UV-finite. The final amplitude is independent of the mass scale $\mu$ in the definition of $B$.

# B Mass Corrections

In this appendix, we give our mass correction for the cross section in (3.17). Specifically, following Ref. [15], we find that the mass correction to (3.17) is

$$
\frac{d\;\Delta\sigma_m}{d\Omega k dk d\Omega_k} = \frac{(p_2^0)^2}{2^9\pi^5 s s'}|A_m|^2, \tag{B.31}
$$

where we have defined

$$
|A_m|^2 \;=\; -\,\frac{e^2 m^2}{(qk)^2} f_0(q-k, p_i). \tag{B.32}
$$

Here, the photon is radiated nearly parallel to $q$, and $f_0$ denotes the non-radiative cross section, summed over all polarizations, with the original $q$ replaced by $q-k$. In the case of Bhabha scattering the Born cross section is proportional to the following invariant summed matrix element squared:

$$
f_B{}^{e^+e^-} \;=\; \frac{2e^4}{t^2}(s^2+u^2). \tag{B.33}
$$

The complete non-radiative cross section for the $\mathrm{O}(\alpha^2)$ single bremsstrahlung mass corrections is then proportional to

$$
f_0{}^{e^+e^-} \;=\; (1+\frac{e^2}{4\pi^2}\mathcal{F}) f_B{}^{e^+e^-} \tag{B.34}
$$

with

$$\mathcal{F}(t) = = \ 2\left(\ln\frac{|t|}{m_e^2} - 1\right). \tag{B.35}$$

From (B.32) it follows that, when summed over all fermion legs, the finite mass terms for the O($\alpha^2$) single bremsstrahlung corrections are given by

$$\begin{aligned}
&|A_{m_e}^{e^+ + e^-}|^2 = \\
&-\frac{2e^6 m_e^2}{(q_1k)^2}\left[1+\frac{e^2}{4\pi^2}\mathcal{F}(-2q_1q_2+2q_2k)\right]\ \frac{(p_1q_1-p_1k)^2+(q_1p_2-p_2k)^2}{(q_1q_2-q_2k)^2}\\
&-\frac{2e^6 m_e^2}{(q_2k)^2}\left[1+\frac{e^2}{4\pi^2}\mathcal{F}(-2q_1q_2-2q_1k)\right]\ \frac{(p_2q_2+p_2k)^2+(p_1q_2+p_1k)^2}{(q_1q_2+q_1k)^2}\\
&-\frac{2e^6 m_e^2}{(p_1k)^2}\left[1+\frac{e^2}{4\pi^2}\mathcal{F}(-2p_1p_2+2p_2k)\right]\ \frac{(p_1q_1-q_1k)^2+(p_1q_2-q_2k)^2}{(p_1p_2-p_2k)^2}\\
&-\frac{2e^6 m_e^2}{(p_2k)^2}\left[1+\frac{e^2}{4\pi^2}\mathcal{F}(-2p_1p_2-2p_1k)\right]\ \frac{(p_2q_2+q_2k)^2+(q_1p_2+q_1k)^2}{(p_1p_2+p_1k)^2}\ .
\end{aligned} \tag{B.36}$$

This completes our appendices.

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